Imperial College
London

Topic 9

Reactance, Impedance and filter circuit

Professor Peter YK Cheung
Dyson School of Design Engineering

URL: www.ee.ic.ac.uk/pcheung/teaching/DE1_EE/
E-mail: p.cheung@imperial.ac.uk

Steady State vs Transient

- An electronic circuit could be responding to either fixed DC signals (such as constant voltage or current sources) or fixed sine, cosine or repetitive signals. The response of the circuit is known as "**steady state response**".
- However, before a circuit reaches steady state, it generally goes through a period of sudden changes, such as being switched from OFF to ON.
- The response of the circuit to these sudden changes is referred to as the "**transient response**".
- If the stimulus to the electronic system is a step function (e.g. it goes from a low voltage level to a high voltage level), the response is known as "**step response**".

Initial and Final values

- RC or CR circuits are called **first-order systems** because their behaviours are determined with a first-order differential equation.
- We can generalize first-order system transient responses in terms of two exponentials:

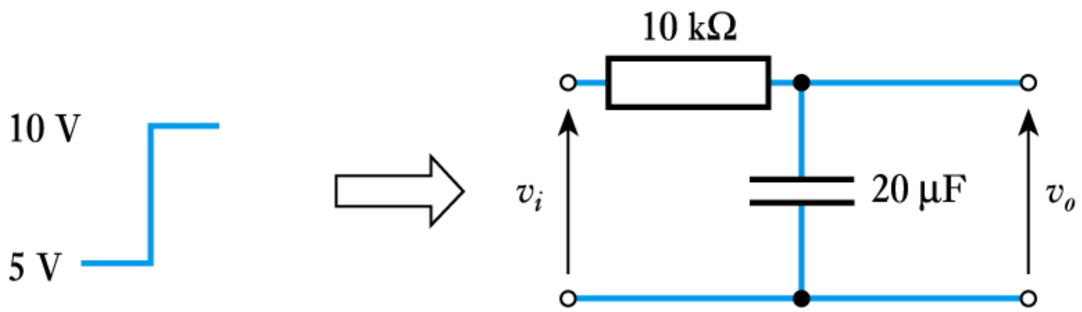
$$v = V_f + (V_i - V_f) \times e^{-\frac{t}{\tau}} \qquad i = I_f + (I_i - I_f) \times e^{-\frac{t}{\tau}}$$

where V_i and I_i are the **initial values** of the voltage and current,
and V_f and I_f are the **final values** of the voltage and current.

- The first terms in these expressions are the **steady-state responses** of the circuit when t ➔ ∞.
- The second terms in these expressions are the **transient responses** of the circuit.
- Together they provide the voltage and current values instantaneously and when t is long. These are the **total responses** of the circuits.

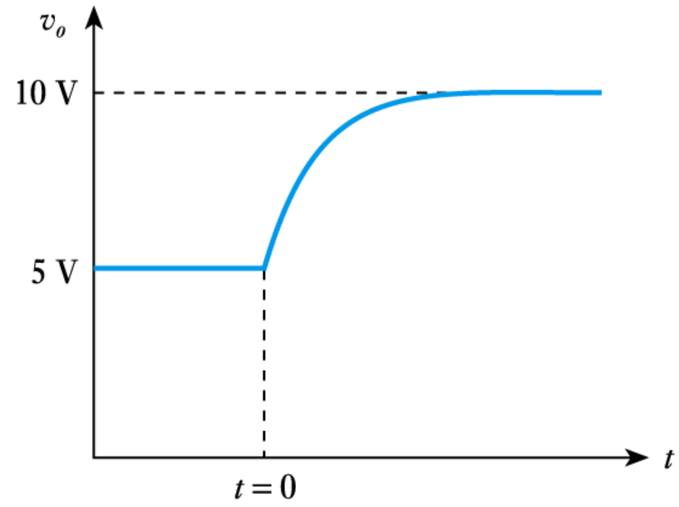
An Example

- The input voltage to the following RC network undergoes a step change from 5 V to 10 V at time t = 0. Derive an expression for the resulting output voltage.

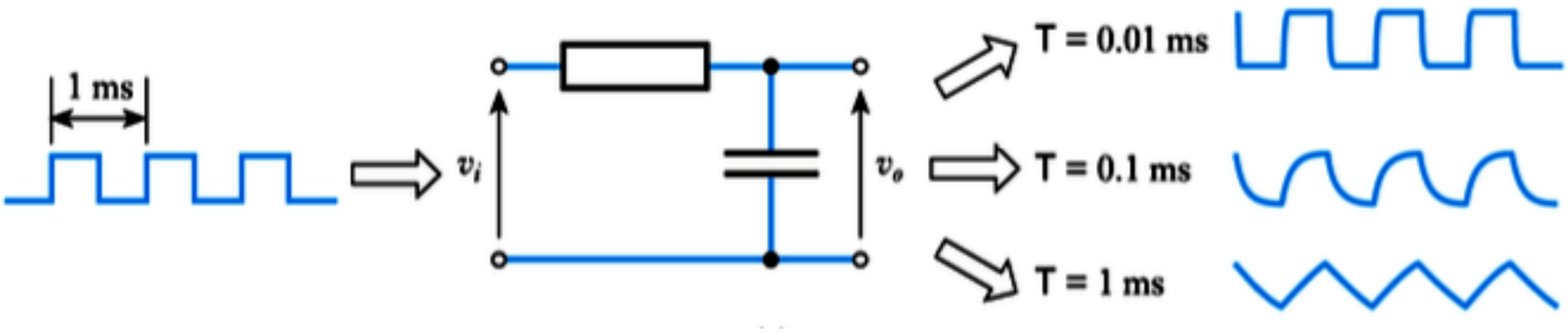


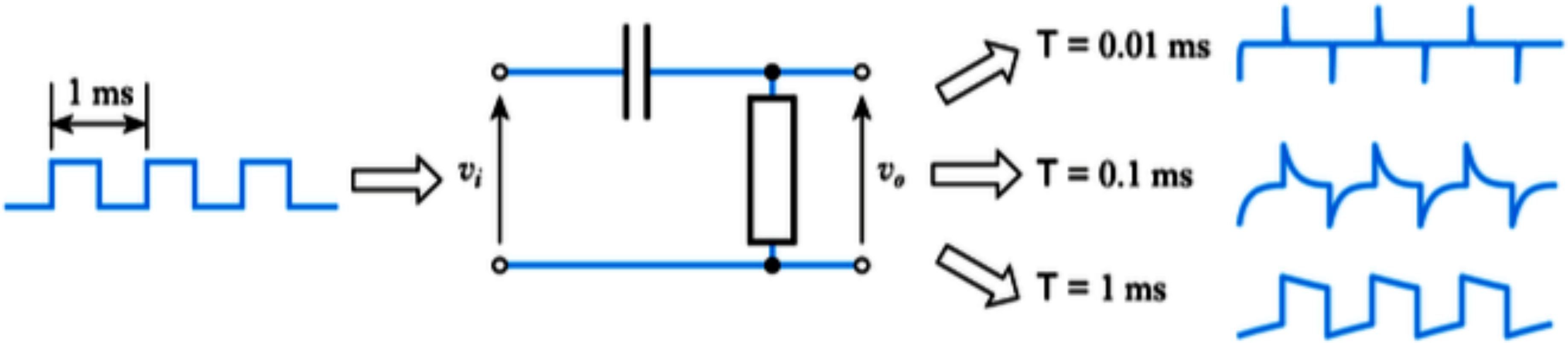
- Here the initial value is 5 V and the final value is 10 V. The time constant of the circuit equals RC = $10 \times 10^3 \times 20 \times 10^{-6}$ = 0.2s. Therefore, from above, for t ≥ 0.

$$\begin{aligned} v &= V_f + (V_i - V_f) \times \mathrm{e}^{-t/\mathrm{T}} \\ &= 10 + (5 - 10) \times \mathrm{e}^{-t/0.2} \\ &= 10 - 5\mathrm{e}^{-t/0.2} \ \text{volts} \end{aligned}$$



Impact of time-constant on pulse responses





Sine and Cosine waves

- We have considered sine wave signals earlier in lectures and labs. We know a sinusoidal signal is given by the equation:

$$v(t) = V_p \sin(2\pi f t + \phi)$$

where V_p is the peak voltage

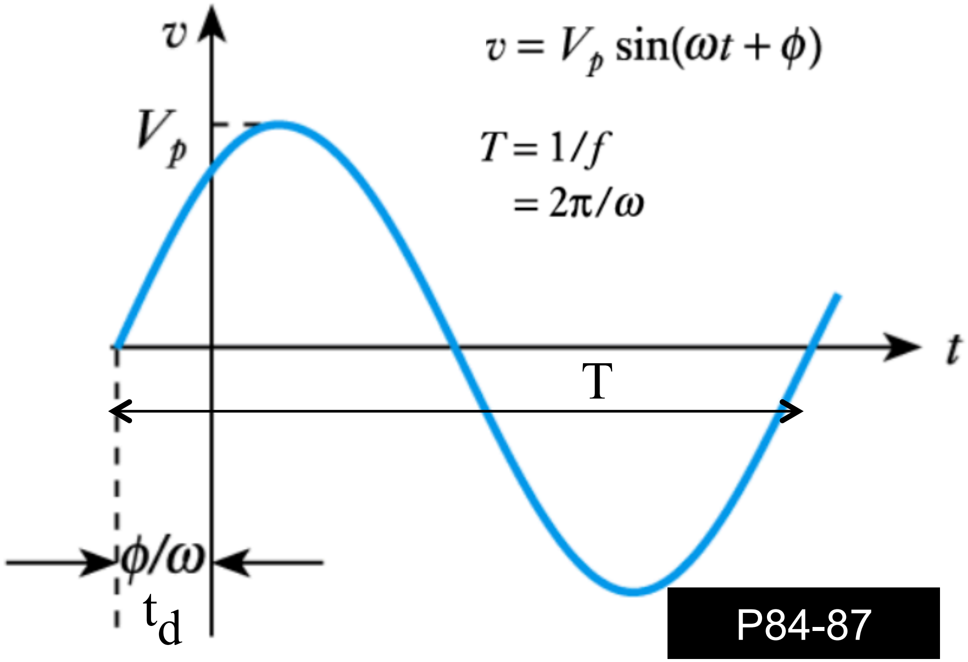
f is the frequency in Hz

Φ is the phase angle, either in radians or in degrees

- We often use ω, the angular frequency in rad/sec, instead of *f*, and $\omega = 2\pi f$

- If Φ is in radians, then the time shift t_d is given by Φ/ω.

- Remember that period T = 1/f and one cycle of a sine wave corresponds to a phase angle of 2π radians or 360 degrees.

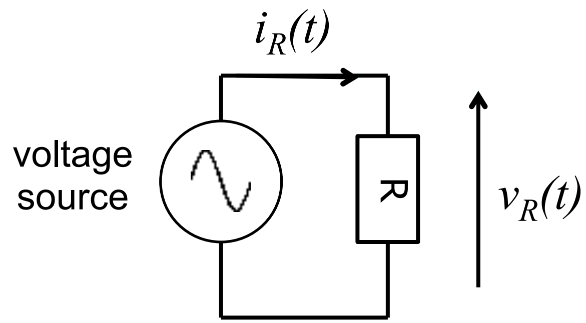


P84-87

Sine wave through a resistor and a capacitor

- Consider the resistor circuit driven by a sinusoidal voltage source:

$$v_R(t) = V_{pk} \sin \omega t$$



- Using Ohm's law, we have:

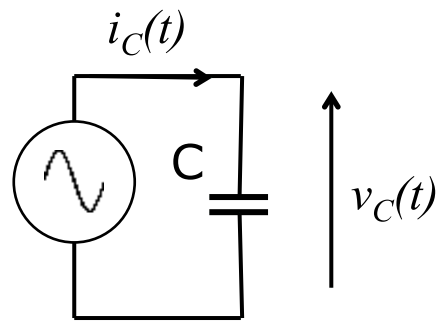
$$i_R(t) = v_R(t) / R = \frac{1}{R} V_{pk} \sin \omega t$$

- Now consider a capacitor driven by the same signal $v_R(t)$.
- Capacitor equation is:

$$i_C(t) = C \frac{dv_C(t)}{dt}$$

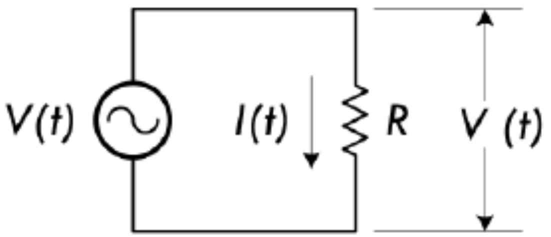
- Hence:

$$v_C(t) = V_{pk} \sin \omega t$$



$$i_C(t) = C \frac{d}{dt}(V_{pk} \sin \omega t) = \omega C \times V_{pk} \cos \omega t$$

AC signal power – RMS voltage



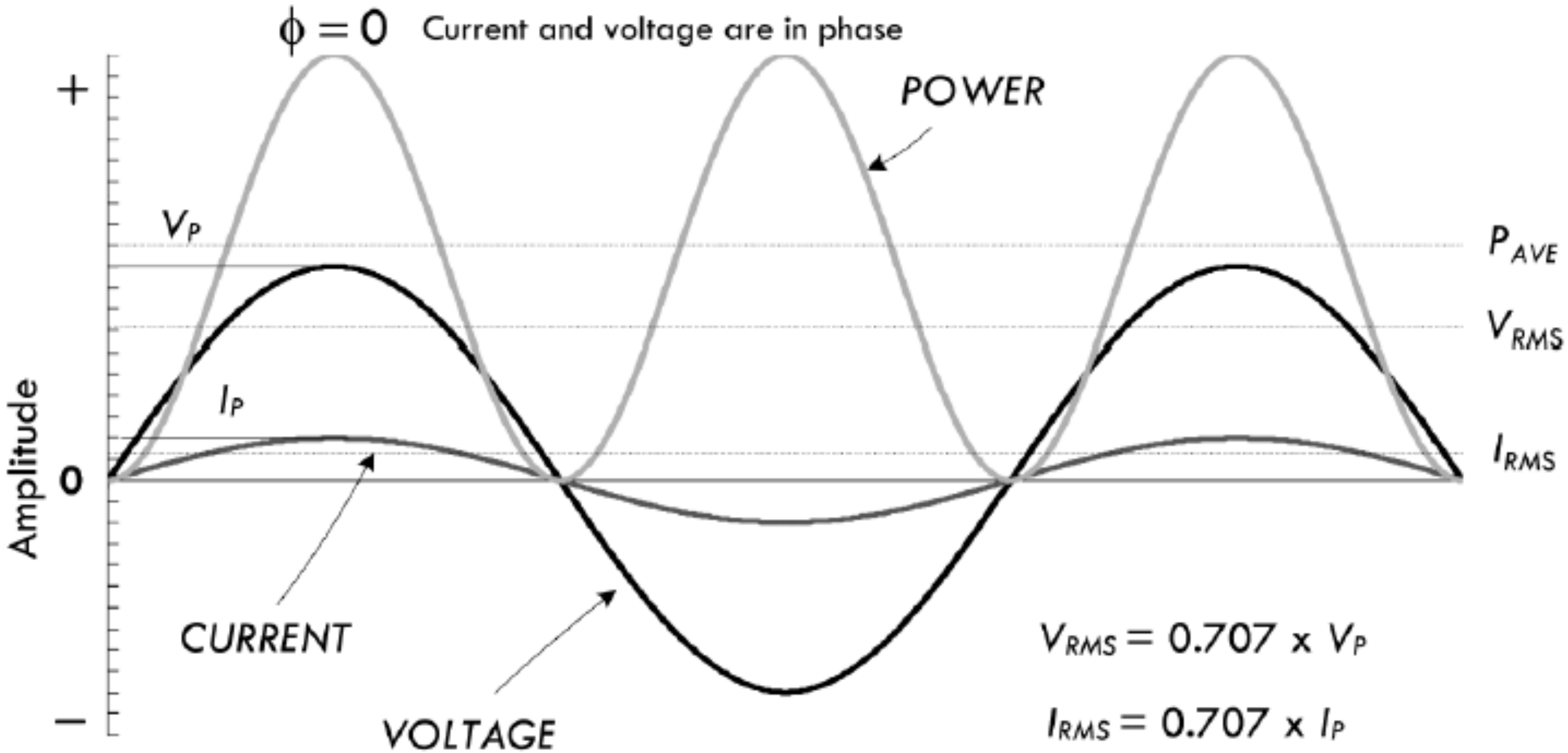
$$V(t) = V_P \sin(2\pi \times f \times t)$$

- According to Ohm's Law,

$$I(t) = \frac{V(t)}{R} = \frac{V_P}{R}\sin(2\pi \times f \times t)$$

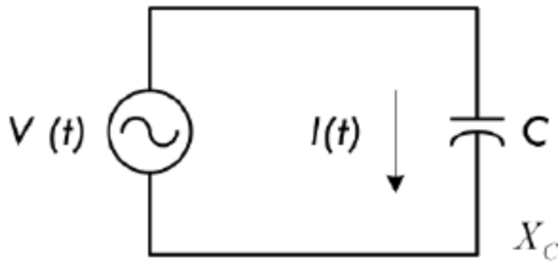
- Since Power = V x I

$$P(t) = \frac{V(t)^2}{R} = \frac{V_P^2}{R}\sin^2(2\pi f t)$$

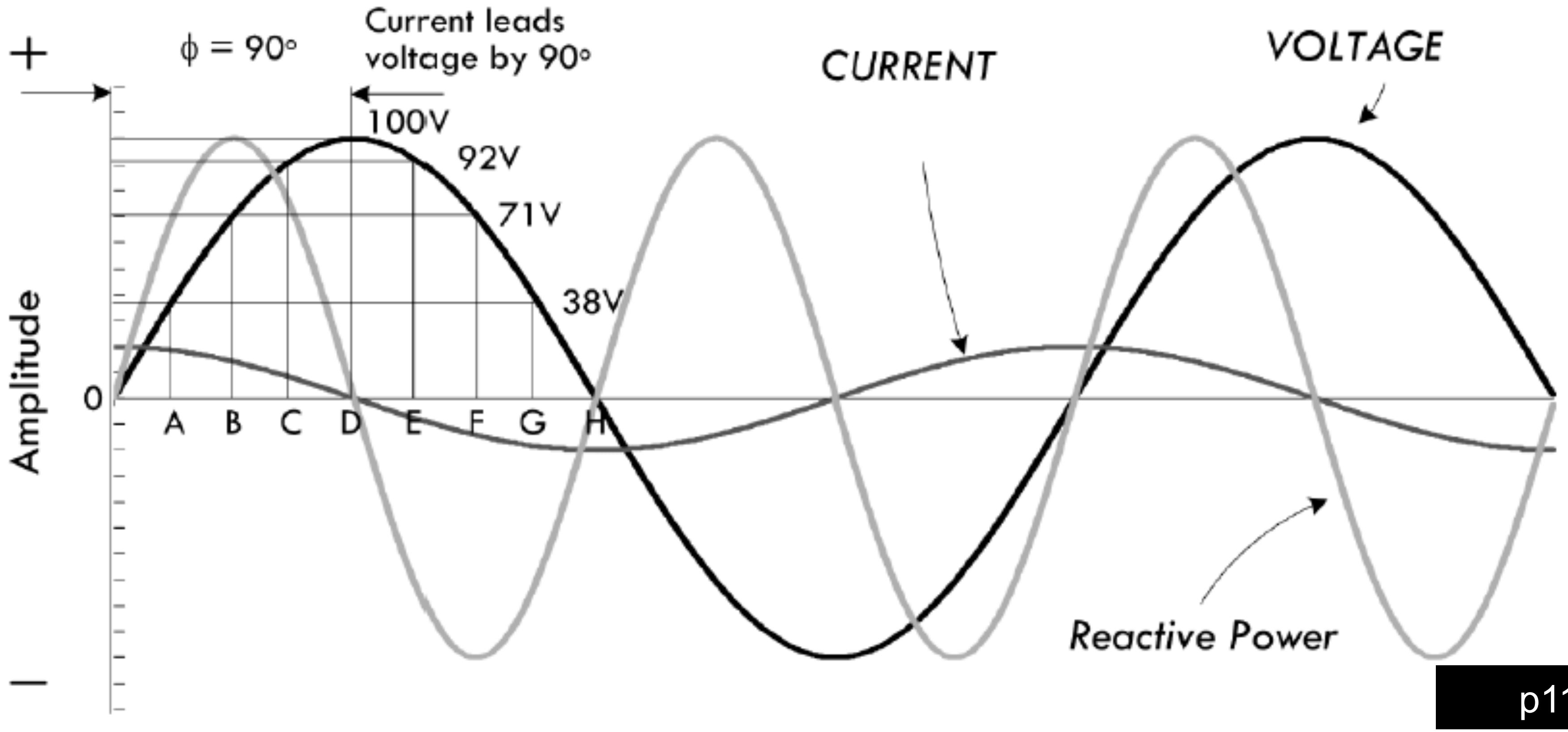


P88

Capacitor stores energy



$$X_C = \frac{1}{2\pi f C}$$



p111

V/I relationships for R and C (peak only)

- Let us ignore phase angle for the moment, and compute **peak voltage** and **peak current** in both cases.
- Resistor (resistance):

$$\frac{v_R(t)_{\max}}{i_R(t)_{\max}} = \frac{(\mathrm{V}_{pk}\sin\omega t)_{\max}}{(V_{pk}\sin\omega t / R)_{\max}} = R$$

- Capacitor (reactance):

$$X_C = \frac{v_C(t)_{\max}}{i_C(t)_{\max}} = \frac{(V_{pk}\sin\omega t)_{\max}}{\omega C(V_{pk}\cos\omega t)_{\max}} = \frac{1}{\omega C}$$

Reactance of Capacitor

- The ratio of voltage to current is a measure of how the component opposes the flow of electricity
- In a resistor, this ratio is the **resistance**
- In capacitors it is termed its **reactance**
- Reactance is given the symbol *X*. Therefore:

$$\text{Reactance of a capacitor, } X_C = \frac{1}{\omega C}$$

- Units of reactance is ohms, same as resistance.
- It can be used in much the same way as resistance: $V = I\,X_C$ $\quad V = I\,X_L$
- Example: A sinusoidal voltage of 5 V peak and 100 Hz is applied across an inductor of 25 mH. What will be the peak current?

$$X_L = \omega L = 2\pi f\,L = 2 \times \pi \times 100 \times 25 \times 10^{-3} = 15.7\Omega$$

$$\text{Therefore} \quad I_L = \frac{V_L}{X_L} = \frac{5}{15.7} = 0.318A \ \ \text{(peak)}$$

Impedance – reactance + phase

- However, remember that peak voltage and peak current in a capacitor happen at different time

$$v_C(t) = V_{pk}\sin\omega t \qquad\qquad i_C(t) = \omega C \times V_{pk}\cos\omega t$$

- Furthermore, for sine signals, capacitor current always **LEADS** capacitor voltage by 90 degrees or $\pi/2$
- To account of the constant phase difference between the two peaks, we define the ratio of the amplitude of capacitor voltage / capacitor current as a complex quantity known as **impedance**, such that:

Impedance: $v_c(t) / i_C(t) = 1 / j\omega C$

- The ratio of voltage to current in a capacitor is now a complex number
- The use of complex number allows us to treat capacitors in a similar way to resistor – all analysis we used for resistors also works here, as long as we use complex number in our calculations

Gain of a Two-port Networks

- While the properties of a pure ***resistance*** are not affected by the frequency of the signal concerned, this is not true of ***reactive*** components.
- We will start with a few basic concepts and then look at the characteristics of simple combinations of resistors and capacitors.

- A two-port network has two ports: an input port, and an output port.
- We can define voltages and currents at the input and output as shown here.
- Then:

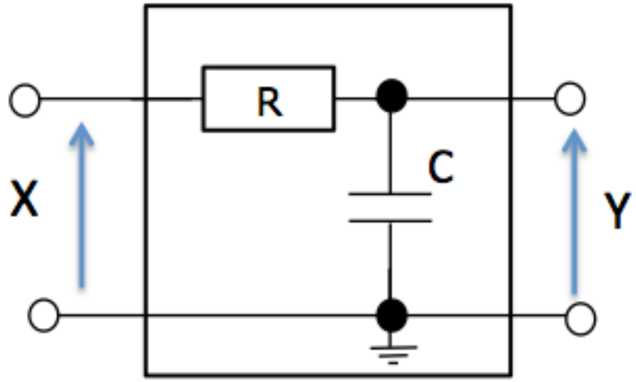
$$\text{voltage gain } (A_v) = \frac{V_O}{V_i}$$

$$\text{current gain } (A_i) = \frac{I_O}{I_i}$$

$$\text{power gain } (A_p) = \frac{P_O}{P_i}$$

Frequency Response

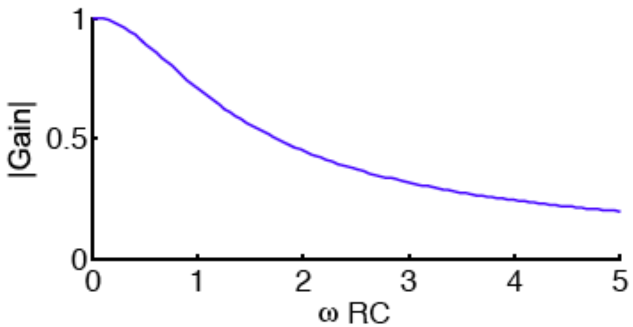
- If $x(t)$ is a sine wave, then $y(t)$ will also be a sine wave but with a different amplitude and phase shift. X is an input phasor and Y is the output phasor.



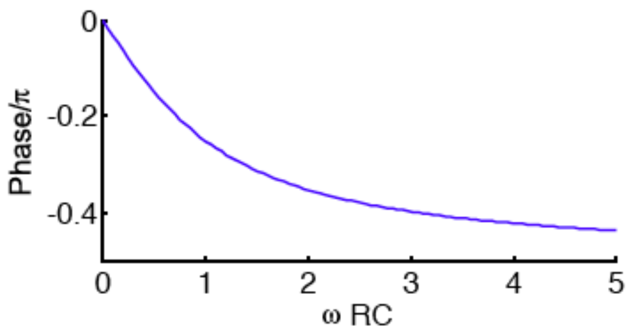
- The *gain* of the circuit is $\frac{Y}{X} = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{j\omega RC+1}$
- This is a complex function of ω so we plot separate graphs for:

Magnitude: $\left|\frac{Y}{X}\right| = \frac{1}{|j\omega RC+1|} = \frac{1}{\sqrt{1+(\omega RC)^2}}$

Phase Shift: $\angle\left(\frac{Y}{X}\right) = -\angle(j\omega RC+1) = -\arctan\left(\frac{\omega RC}{1}\right)$



Magnitude Response

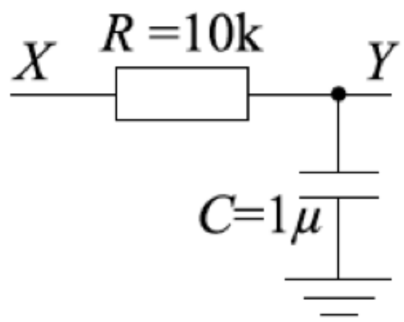


Phase Response

Sine Wave Response

$$RC = 10\text{ ms}$$

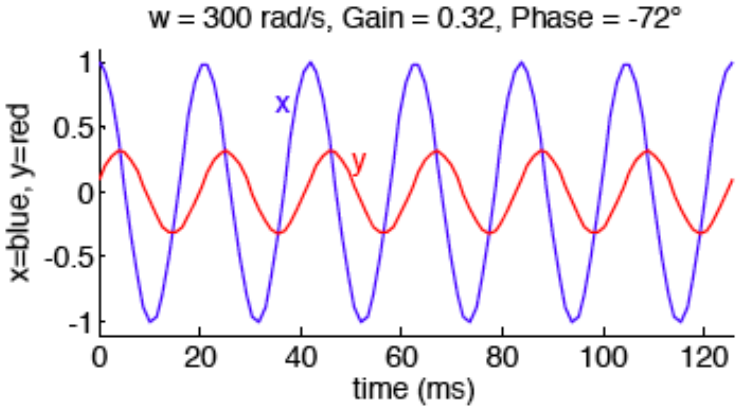
$$\frac{Y}{X} = \frac{1}{j\omega RC+1} = \frac{1}{0.01j\omega+1}$$

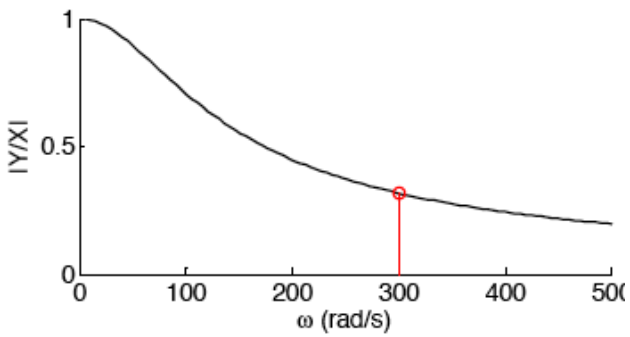


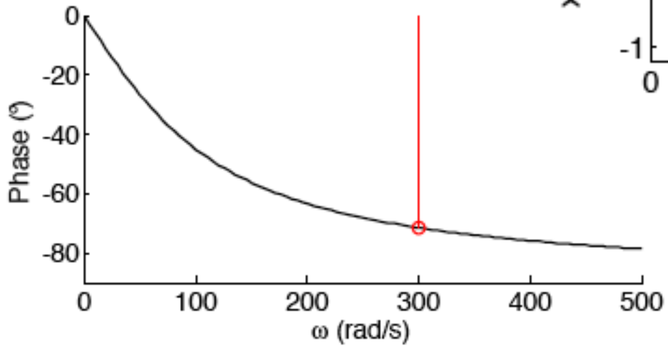
$$\omega = 50 \Rightarrow \frac{Y}{X} = 0.89\angle -27^\circ$$

$$\omega = 100 \Rightarrow \frac{Y}{X} = 0.71\angle -45^\circ$$

$$\omega = 300 \Rightarrow \frac{Y}{X} = 0.32\angle -72^\circ$$



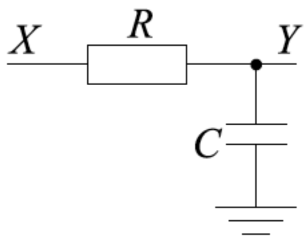




- The output, *y(t)*, lags the input, *x(t)*, by up to *90*°.

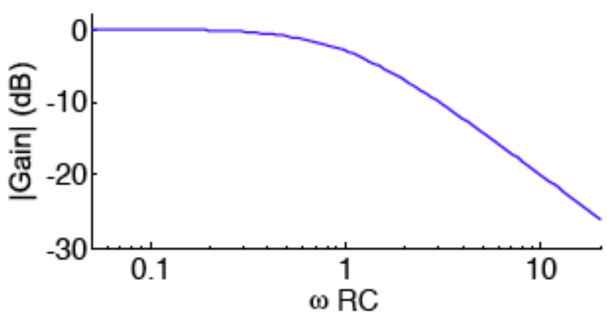
Logarithmic axes

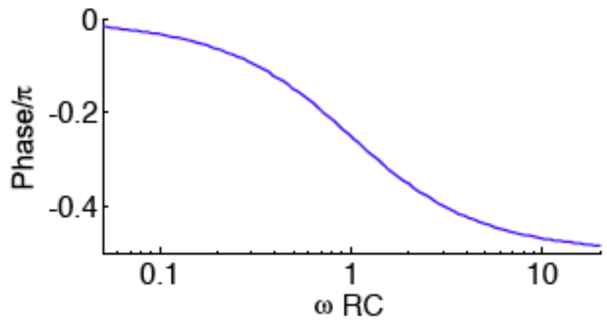
- We usually use logarithmic axes for frequency and gain (but not phase) because % differences are more significant than absolute differences.
- E.g. 5 kHz versus 5.005 kHz is less significant than 10Hz versus 15Hz even though both differences equal 5Hz.
- Logarithmic voltage ratios are specified in *decibels* (dB) = $20 \log_{10} |V_2 / V_1|$.



- Common voltage ratios:

$\frac{\lvert V_2 \rvert}{\lvert V_1 \rvert}$	0.1	0.5	$\sqrt{0.5}$	1	$\sqrt{2}$	2	10	100
dB	−20	-6	-3	0	3	6	20	40





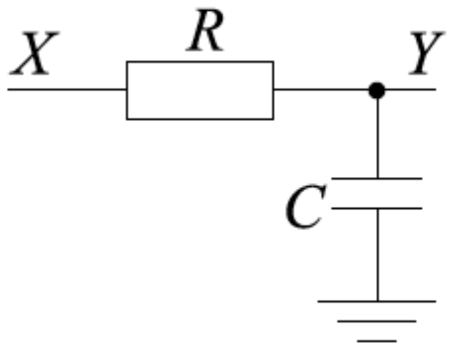
Note that 0 does not exist on a log axis and so the starting point of the axis is arbitrary.

Note: $P \propto V^2 \Rightarrow$ decibel <u>power</u> ratios are given by **10** $\log_{10} (P_2 / P_1)$

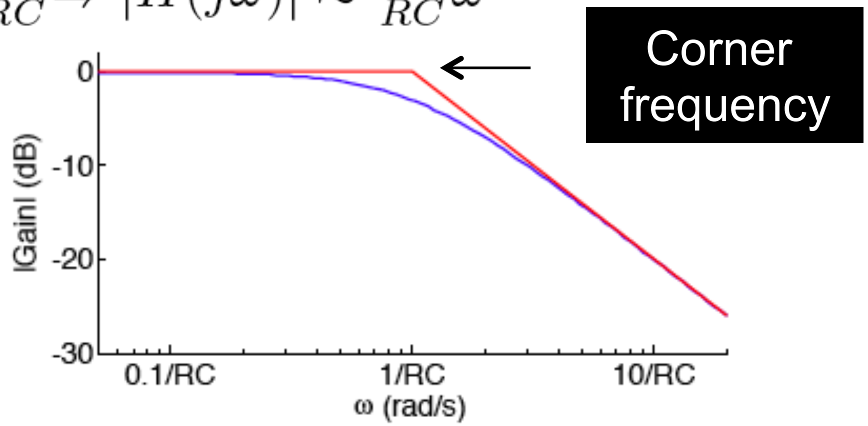
P204-206

Straight Line Approximations

- Key idea: $(aj\omega + b) \approx \begin{cases} aj\omega & \text{for } |a\omega| \gg |b| \\ b & \text{for } |a\omega| \ll |b| \end{cases}$



- Gain: $H(j\omega) = \frac{1}{j\omega RC + 1}$
- Low frequencies: $(\omega \ll \frac{1}{RC})$: $H(j\omega) \approx 1 \Rightarrow |H(j\omega)| \approx 1$
- High frequencies: $(\omega \gg \frac{1}{RC})$: $H(j\omega) \approx \frac{1}{j\omega RC} \Rightarrow |H(j\omega)| \approx \frac{1}{RC}\omega^{-1}$
- Approximate the magnitude response as two straight lines intersecting at the corner frequency, ωc = 1/RC .



- At the corner frequency:

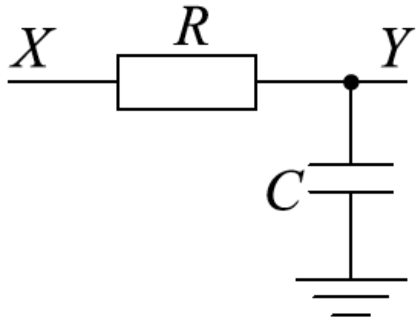
  (a) the gradient changes by −1 (= −6 dB/octave = −20 dB/decade).

  (b) $|H(j\omega_c)| = \left|\frac{1}{1+j}\right| = 1/\sqrt{2} = -3$ dB (worst-case error).

A linear factor $(aj\omega + b)$ has a corner frequency of $\omega_c = |b/a|$.

1^{st} Order Low Pass Filter

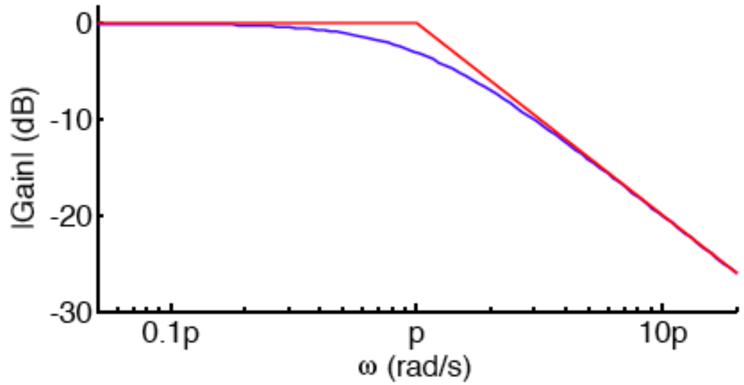
$$\frac{Y}{X} = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{j\omega RC+1} = \frac{1}{\frac{j\omega}{p}+1}$$



- Corner frequency: $p = \left|\frac{b}{a}\right| = \frac{1}{RC}$
- Asymptotes: 1 and $\frac{p}{j\omega}$

  Very low ω: Capacitor = open circuit

  Very high ω: Capacitor = short circuit



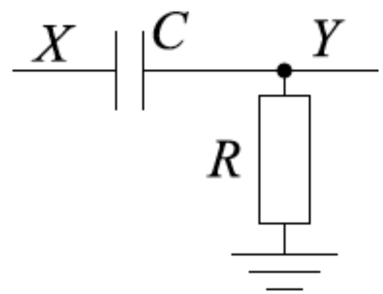
- A low-pass filter because it allows low frequencies to pass but ***attenuates*** (makes smaller) high frequencies.

- The ***order*** of a filter: highest power of $j\omega$ in the denominator.
- Almost always equals the total number of L and/or C.

P210-213

High Pass Filter

$$\frac{Y}{X} = \frac{R}{R + {}^{1}/_{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1}$$



- Corner frequency: $p = \frac{1}{RC}$

- Asymptotes: $j\omega RC$ and 1

Very low ω:

Capacitor = open circuit

Gain = 0

Very high ω:

Capacitor short circuit

Gain = 1